

## WEAK IDENTITIES IN FINITELY GENERATED GROUPS

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ABSTRACT. In this article we introduce the notion of weak identities in a group and study their properties. We show that weak identities have some similar properties to ordinary ones. We use this notion to prove that any finitely generated solvable discriminating group is abelian, which answers a question raised in [3].

## 1. INTRODUCTION

In this article we introduce the notion of weak identities in a group and study their properties. An element  $f(g_1, \dots, g_k)$  in the free group on  $k$  generators is called a weak identity in group  $G$  if there exists  $N$ , such that for any  $N$   $k$ -tuples  $(h_{i1}, \dots, h_{ik})_{i=1, N}$  of elements in the group  $G$  such that any two elements in different  $k$ -tuples commute, then  $f$  evaluated at one  $k$ -tuple gives identity<sup>1</sup>.

First, we show that the set of weak identities in a given group  $G$  form a verbal subgroup. This result shows that weak identities are similar to the ordinary ones, but there are some substantial differences – the main ones are:

- the free group has many nontrivial weak identities;
- the class of groups satisfying a given weak identity is not closed under taking homomorphic images.

We show that a very large class of groups satisfy the weak identity  $[g_1, g_2]$  – this class includes all linear groups. Next, we study weak identities modulo a verbal subgroup, and use them to construct a relation on all verbal subgroups of a free group. We also introduce weak\* identities, since the above relation is not transitive. Finally, we use the notion of weak identity to study discriminating groups and answer two questions raised in [3].

The paper is organized as follows: The notion of weak identities is defined in section 2. In section 3, we discuss the notion of a group having a bounded centralizer sequence and show the connection with weak identities. In section 4, we showed that the notion of weak identities is not transitive and define weak\* identities in order to address this problem. In section 5, we investigate weak\* identities in finitely generated meta abelian groups. In section 6, we apply the results from the previous sections to discriminating groups and give answers to the Questions 2D and 3D from [3] - we prove that: every linear discriminating group is abelian; and that every finitely generated solvable discriminating group is free abelian. Finally, in section 7 we pose some open questions concerning the notion of weak identities.

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<sup>1</sup>The main differences between the weak identities and the ordinary ones come from the fact that it is only known that  $f$  vanishes on some  $k$ -tuple but it is not known on which one.

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## 2. WEAK IDENTITIES

In this section we define the notion of weak identities in a group. They have similar properties to ordinary identities – see theorems 2.5 and 4.5; however there are also few substantial differences – see remarks 2.6 and 2.8.

*Definition 2.1.* Let  $\mathfrak{F}$  be the free group on countably many generators  $g_i$ , for  $i \in \mathbb{N}$ . A subgroup  $\mathfrak{H}$  in  $\mathfrak{F}$  is called a  $T$ -subgroup or verbal subgroup (denoted  $\mathfrak{H} \trianglelefteq_T \mathfrak{F}$ ) if it is preserved by all endomorphisms of the group  $\mathfrak{F}$ . For a set  $S \subset \mathfrak{F}$ , we will denote by  $\langle S \rangle_T$  the minimal  $T$ -subgroup which contains  $S$ .

*Definition 2.2.* Let  $G$  be an abstract group. We say that the set  $S \subset G$  is a set of weak identities in the group  $G$ , if there exists an integer  $N$  such that for any elements  $s_k \in S$ , for  $k = 1, \dots, N$ , and any homomorphism

$$\rho : \mathfrak{F}^{\times N} = \mathfrak{F} \times \mathfrak{F} \times \dots \times \mathfrak{F} \rightarrow G,$$

there exists an index  $k$  between 1 and  $N$  such that

$$\rho(i_k(s_k)) = 1,$$

where  $i_k$  denotes the inclusion of  $\mathfrak{F}$  in  $\mathfrak{F}^{\times N}$  at the  $k$ -th component. The number  $N$  is called the height of the set  $S$  of weak identities.

An element  $f \in \mathfrak{F}$  is called a weak identity in  $G$ , if the set  $\{f\}$  is a set of weak identities.

*Corollary 2.3.* Let  $S_i$  be a finite collection of sets, such that  $S_i$  is a set of weak identities in  $G$  of height  $n_i$ , for each  $i$ . Then the union  $\cup S_i$  is also a set of weak identities of height at most  $\sum n_i$ .

*Example 2.4.* The element  $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$  is a weak identity of height 2 in the free group  $\mathfrak{F}$ , but it is not an ordinary identity. In order to prove this we need to show that for any homomorphism  $\rho : \mathfrak{F}^{\times 2} \rightarrow \mathfrak{F}$ , we have that  $\rho(i_1[g_1, g_2]) = 1$  or  $\rho(i_2[g_1, g_2]) = 1$ . Let us assume that  $\rho(i_1([g_1, g_2])) \neq 1$ . Then the centralizer of this element in the free group  $\mathfrak{F}$  is an infinite abelian group  $H$ . The elements  $\rho(i_2(g_1))$  and  $\rho(i_2(g_2))$  lie in  $H$  because they commute with  $\rho(i_1([g_1, g_2]))$ . This implies that their commutator is 1. The above argument shows that at least one of the elements  $\rho(i_1([g_1, g_2]))$  or  $\rho(i_2([g_1, g_2]))$  is 1, i.e.,  $[g_1, g_2]$  is a weak identity in the free group  $\mathfrak{F}$  of height 2. This example will be generalized in section 3.

The next theorem states that in order to show that a  $T$ -subgroup consists of weak identities, it is enough to verify that the set of its generators are form a set of weak identities.

**Theorem 2.5.** *If  $S$  is a set of weak identities in the group  $G$ , then the  $T$ -subgroup  $\mathfrak{S} = \langle S \rangle_T \trianglelefteq_T \mathfrak{F}$  generated by the set  $S$  is also a set of weak identities.*

*Proof.* Any element  $g$  in the  $T$ -subgroup  $\mathfrak{S}$ , generated by  $S$ , can be written as the product

$$g = \prod_{k=i}^n \phi_i(s_i),$$

where  $s_i \in S$  and  $\phi_i : \mathfrak{F} \rightarrow \mathfrak{F}$  are endomorphisms.

Suppose that we have  $N$  elements  $g_j \in \mathfrak{S}$  and a homomorphism  $\rho : \mathfrak{F}^{\times N} \rightarrow G$ . We can write any of the elements  $g_j$  in the form

$$g_j = \prod_{i=1}^{n_j} \phi_{i,j}(s_{i,j}).$$

Let  $i(j)$  be the index between 1 and  $n_j$  such that the length of  $\rho(i_j(\phi_{i,j}(s_{i,j})))$  is maximal.<sup>2</sup>

Now consider the homomorphism  $\tilde{\rho} : \mathfrak{F}^{\times N} \rightarrow G$  defined by

$$\tilde{\rho} = \rho \circ (\phi_{i(1),1} \times \cdots \times \phi_{i(N),N}).$$

By definition we have that  $\tilde{\rho}(i_j(s_{i(j),j})) = \rho(i_j(\phi_{i(j),j}(s_{i(j),j})))$ .

Using the homomorphism  $\tilde{\rho}$  and fact that  $S$  is a set of weak identities in  $G$  (by construction we have  $s_{i(j),j} \in S$ , for all  $j$ ), we know that there exists  $j$  such that  $\tilde{\rho}(i_j(s_{i(j),j})) = 1$ .

By the definition of  $i(j)$ , we have that the length of  $\rho(i_j(\phi_{i(j),j}(s_{i(j),j})))$  is bigger than or equal to the length of  $\rho(i_j(\phi_{i,j}(s_{i,j})))$  for any  $i$ . However, the first element is identity and has length zero, therefore the lengths of all elements  $\rho(i_j(\phi_{i,j}(s_{i,j})))$  are 0, i.e., all of them are equal to the identity in  $G$ . This shows that

$$\rho(i_j(g_j)) = \prod \rho(i_j(\phi_{i,j}(s_{i,j}))) = 1,$$

which shows that the subgroup  $\mathfrak{S}$  is a set of weak identities in  $G$ .  $\square$

*Remark 2.6.* Let us fix a group  $G$ . Denote by  $\mathfrak{w}\mathfrak{Id}(G)$  the set of all elements  $f$  in  $\mathfrak{F}$ , such that the set  $\{f\}$  is a set of weak identities in  $G$ . By the previous theorem, this is a  $T$ -subgroup in  $\mathfrak{F}$  which is called the group of weak identities in  $G$ . Note that this theorem does not imply that  $\mathfrak{w}\mathfrak{Id}(G)$  is a set of weak identities in  $G$ . However, any finitely generated<sup>3</sup>  $T$ -subgroup  $\mathfrak{H}$  of  $\mathfrak{w}\mathfrak{Id}(G)$  is a set of weak identities.

*Example 2.7.* If the group  $G$  is finite, then  $\mathfrak{w}\mathfrak{Id}(G) = \mathfrak{F}' \cdot \mathfrak{F}^n$ , where  $n$  is the minimal number such that  $g^n = 1$  for any  $g \in G$ .

First, let us show that  $[g_1, g_2]$  is a weak identity in  $G$ . Suppose that it is not. Then for any  $N$ , there exist elements  $g_i$  and  $h_i$ , for  $1 \leq i \leq N$ , in the group  $G$ , such that

$$[g_i, g_j] = 1 \quad [h_i, h_j] = 1 \quad [g_i, h_j] = 1, \text{ iff } i \neq j \quad [g_i, h_i] \neq 1.$$

Let us define the subgroups  $P_i$  of  $G$  using these elements  $g_i$  by

$$P_i = \{g \in G \mid [g, g_j] = 1 \text{ for } j < i\},$$

It is easy to check that  $h_i \in P_{i-1} \setminus P_i$ , i.e.  $P_i$  form a strictly descending sequence of subgroups in  $G$ , which is impossible if  $N > \log_2 |G|$ . This contradiction shows that  $[g_1, g_2]$  is a weak identity in any finite group  $G$  (of height at most  $\log_2 |G|$ ).

Since the element  $g^n$  is an identity in  $G$ , it is also a weak identity (of height 1). Therefore, the group  $\mathfrak{w}\mathfrak{Id}(G)$  contains the subgroup  $\mathfrak{F}' \cdot \mathfrak{F}^n$ . If we assume that the inclusion is strict, then the group  $\mathfrak{w}\mathfrak{Id}(G)$  would contain the element  $g^k$  for some  $0 < k < n$ . The last element is not an identity in  $G$ . It can be shown that it is not also a weak identity. This proves that  $\mathfrak{w}\mathfrak{Id}(G) = \mathfrak{F}' \cdot \mathfrak{F}^n$ .

<sup>2</sup>In order to have a notion of length of an element in  $G$ , we need to fix a generating set of  $G$ .

<sup>3</sup>as  $T$ -subgroup

*Remark 2.8.* Let  $\mathfrak{H}$  be a  $T$ -subgroup. Denote by  $\mathfrak{wVar}(\mathfrak{H})$  the class of all groups  $G$  such that any element in  $\mathfrak{H}$  is a weak identity in  $G^4$ . It can be shown that the class  $\mathfrak{wVar}(\mathfrak{H})$  is closed under taking subgroups and taking finite Cartesian products, but not infinite products. In general is not closed under taking homomorphism images, although it is closed under taking some kinds of restricted homomorphic images (see open problem 7.3). This is one important difference between weak identities and the ordinary ones, since it implies that there are no universal objects in the class  $\mathfrak{wVar}(\mathfrak{H})$ .

### 3. WEAK IDENTITIES IN LINEAR GROUPS

In this section we show that a large class of groups lie in  $\mathfrak{wVar}(\mathfrak{F}')$ . This generalizes examples 2.4 and 2.7.

*Definition 3.1.* A group  $G$  is said to have bounded centralizer sequences if there exists an integer  $N$ , such that any strictly increasing sequence of stabilizers of sets of mutually commuting elements has length less than  $N$ . That is, for any sequence of subsets  $\{P_i\}_{i=1}^N$  of  $G$  such that

$$P_1 \subset P_2 \subset \cdots \subset P_N,$$

and  $[P_i, P_i] = 1$  for any  $i$ , we have that the sequence of their centralizers

$$\text{Cen}(P_1) \supset \text{Cen}(P_2) \supset \cdots \supset \text{Cen}(P_N)$$

is not strictly decreasing.

*Example 3.2.* Any finite group has bounded centralizer sequences and the same is true for any free group.

**Lemma 3.3.** *If the group  $G$  has bounded centralizer sequences, then  $[g_1, g_2]$  is a weak identity in  $G$ .*

*Proof.* Suppose that  $[g_1, g_2]$  is not a weak identity in  $G$ . Then for any  $N$  there exist elements  $g_i$  and  $h_i$ , for  $1 \leq i \leq N$  in the group  $G$ , such that

$$[g_i, g_j] = 1, \quad [h_i, h_j] = 1, \quad [g_i, h_j] = 1 \text{ iff } i \neq j, \quad [g_i, h_i] \neq 1.$$

The centralizers of the sets  $P_i = \{g_1, \dots, g_i\}$  of mutually commuting elements, form a strictly descending sequence, because  $h_i \in \text{Cen}(P_{i-1}) \setminus \text{Cen}(P_i)$ . This contradicts the assumption that the group  $G$  has bounded centralizer sequences. Therefore,  $[g_1, g_2]$  is a weak identity in the group  $G$  of height equal to the maximal length of an increasing sequence of centralizers in  $G$ .  $\square$

**Theorem 3.4.** *If  $G$  is a linear group, then  $\mathfrak{F}'$  consists of weak identities in  $G$ .*

*Proof.* By lemma 3.3 it is enough to show that any linear group has a bounded sequence of centralizers. The centralizer of a subset  $P$  of the linear group  $GL_n$  is the same as the centralizer of the linear span of  $P$  in the matrix algebra  $M_n$ . But in the matrix algebra  $M_n$  any strictly increasing sequence of sub-spaces has bounded length. Therefore, any strictly decreasing sequence of centralizers has a bounded length.  $\square$

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<sup>4</sup>This is the same as saying that any finite subset is a set of weak identities in the group  $G$ .

## 4. WEAK\* IDENTITIES

It is also possible to define weak identities modulo some  $T$ -subgroup  $\mathfrak{H}$ . In order to do this, we need to define the normal subgroup  $\mathfrak{H}(G)$  of  $G$ .

*Definition 4.1.* Let  $\mathfrak{H} \leq_T \mathfrak{F}$  be a  $T$ -subgroup, and let  $G$  be a group. Denote by

$$\mathfrak{H}(G) = \{\pi(h) \mid h \in \mathfrak{H}, \pi : \mathfrak{F} \rightarrow G\}$$

the subgroup of  $G$  consisting of all elements which are the images of elements in  $\mathfrak{H}$  under homomorphisms from  $\mathfrak{F}$  to  $G$ .

Let us fix a finitely generated group  $G$ . For the rest of this section all the identities we consider are in the group  $G$ , unless stated otherwise.

*Definition 4.2.* A set  $S$  is said to be a set of weak identities (in the group  $G$ ) modulo the  $T$ -subgroup  $\mathfrak{H}$ , denoted  $S \equiv_w 1(\text{mod } \mathfrak{H})$ , if  $S$  is a set of weak identities in the group  $G/\mathfrak{H}(G)$ . This gives rise to a relation on verbal subgroups of the free group: we say that  $\mathfrak{H}_1 \preceq_{wG} \mathfrak{H}_2$  iff  $\mathfrak{H}_1 \equiv_w 1(\text{mod } \mathfrak{H}_2)$  in the group  $G$ . In the case of ordinary identities the above relation comes from the inclusion of the corresponding subgroups, i.e.,  $\mathfrak{H}_1 \preceq_G \mathfrak{H}_2$  iff  $\mathfrak{H}_1(G) \subseteq \mathfrak{H}_2(G)$ .

*Remark 4.3.* Note that if one takes the class of groups  $G$  such that  $S$  is a set of weak identities modulo  $\mathfrak{H}$ , this class is not closed under taking subgroups. The same is true if one considers ordinary identities.

In order to state Theorem 4.5, we need to describe one construction of verbal subgroups.

*Definition 4.4.* Let  $\mathfrak{F}_n$  be the free group generated by  $g_k$ , for  $k = 1, \dots, n$ , and let  $f \in \mathfrak{F}_n$  be an element in it. For a  $T$ -subgroup  $\mathfrak{H}$  and an index  $1 \leq i \leq n$ , denote by  $\langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T$  the  $T$ -subgroup generated by the set

$$(1) \quad \{\rho(f) \mid \rho : \mathfrak{F}_n \rightarrow \mathfrak{F}, \rho(g_j) = g_j, \text{ for } j \neq i, \rho(g_i) \in \mathfrak{H}\},$$

i.e., all the elements which can be obtained from the element  $f$  by substituting a word from  $\mathfrak{H}$  in the place of  $g_i$ .

Similarly, if  $\mathfrak{H}_i$  are  $T$ -subgroups, for  $i = 1, \dots, n$ , then by  $\langle f_{|g_i \rightarrow \mathfrak{H}_i} \rangle_T$  we will denote the  $T$ -subgroup in  $\mathfrak{F}$  generated by

$$(2) \quad \{\rho(f) \mid \rho : \mathfrak{F}_n \rightarrow \mathfrak{F}, \rho(g_i) \in \mathfrak{H}_i \text{ for all } i\}.$$

**Theorem 4.5.** *a) Let  $f \in \mathfrak{F}_n$  and let  $\mathfrak{S}$  and  $\mathfrak{H}$  be  $T$ -subgroups in  $\mathfrak{F}$ . If  $\mathfrak{S} \equiv_w 1(\text{mod } \mathfrak{H})$ , then  $\langle f_{|g_i \rightarrow \mathfrak{S}} \rangle_T \equiv_w 1(\text{mod } \langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T)$ .*

*b) Let  $f \in \mathfrak{F}_n$  and let  $\mathfrak{S}_i$  and  $\mathfrak{H}_i$  be  $T$ -subgroups in  $\mathfrak{F}$ . If  $\mathfrak{S}_i \equiv_w 1(\text{mod } \mathfrak{H}_i)$  for every  $i$ , then  $\langle f_{|g_i \rightarrow \mathfrak{S}_i} \rangle_T \equiv_w 1(\text{mod } \langle f_{|g_i \rightarrow \mathfrak{H}_i} \rangle_T)$ .*

*Proof.* a) Let  $N$  be the height of the set  $\mathfrak{S}$  of weak identities modulo  $\mathfrak{H}$ . First, we will show that the set (1) of generators of the  $T$ -subgroup  $\langle f_{|g_i \rightarrow \mathfrak{S}} \rangle_T$  form a set of weak identities modulo  $\langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T$  of height  $N$ . Suppose that

$$a_k = \pi_k(f), \text{ where } \pi_k : \mathfrak{F}_n \rightarrow \mathfrak{F}, \pi_k(g_j) = g_j, \text{ for } j \neq i, \pi_k(g_i) \in \mathfrak{S},$$

are elements of type (1) and that  $\rho : \mathfrak{F}^{\times N} \rightarrow G$  is a homomorphism. Since  $\mathfrak{S} \equiv_w 1(\text{mod } \mathfrak{H})$  and  $\pi_j(g_i) \in \mathfrak{S}$ , there exists an index  $j$  such that  $\rho(i_j(\pi_j(g_i))) \in \mathfrak{H}(G)$ . Therefore, there exists an element  $h \in \mathfrak{H}$  and a homomorphism  $\bar{\pi} : \mathfrak{F} \rightarrow G$  such that

$$\rho(i_j(\pi_j(g_i))) = \bar{\pi}(h).$$

Without loss of generality, we may assume that the element  $h$  does not depend on the letters  $g_k$  for  $k \leq n$ . Let us define  $\tilde{\pi} : \mathfrak{F}_n \rightarrow \mathfrak{F}$  to be a homomorphism which sends  $g_k$  to  $g_k$  for  $k \neq i$  and  $\tilde{\pi}(g_i) = h$ . Also define  $\tilde{\rho} : \mathfrak{F} \rightarrow G$  by

$$\tilde{\rho}(g_i) = \rho(i_j(g_i)), \text{ for } i \leq n \quad \text{and} \quad \tilde{\rho}(g_i) = \tilde{\pi}(g_i), \text{ for } i > n.$$

Then we have that

$$\tilde{\rho}(\tilde{\pi}(f)) = \rho(i_j(a_j)).$$

However,  $\tilde{\pi}(f) \in \langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T$  because  $\tilde{\pi}(g_i) \in \mathfrak{H}$ . This shows that

$$\rho(i_j(a_j)) \in \langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T(G).$$

This proves that the generators of the  $T$ -subgroup  $\langle f_{|g_i \rightarrow \mathfrak{S}} \rangle_T$  are weak identities of height  $N$  in  $G/\langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T(G)$ . Finally, we can use lemma 2.5 to show that the whole  $T$ -subgroup consist of weak identities.

b) Let  $N_i$  be the heights of the sets  $\mathfrak{S}_i$  as weak identities modulo  $\mathfrak{H}_i$ . Suppose that

$$a_k = \pi_k(f), \text{ where } \pi_k : \mathfrak{F}_n \rightarrow \mathfrak{F}, \pi_k(g_i) \in \mathfrak{S}_i,$$

are elements of type (2), where  $N = \sum N_i$ . Also suppose that  $\rho : \mathfrak{F}^{\times N} \rightarrow G$  is a homomorphism. Using that  $\mathfrak{S}_i \equiv_w 1 \pmod{\mathfrak{H}_i}$  and  $\pi_j(g_i) \in \mathfrak{S}_i$  and that  $N$  is big enough, we can show that there exists an index  $j$  such that  $\rho(i_j(\pi_j(g_i))) \in \mathfrak{H}_i(G)$ , for every  $i$ . Now we can repeat the proof of part a) to show that

$$\rho(i_j(a_j)) \in \langle f_{g_i \rightarrow \mathfrak{H}_i} \rangle_T(G).$$

□

There is one substantial difference between weak identities and ordinary ones: if we have three verbal subgroups  $\mathfrak{H}_1$ ,  $\mathfrak{H}_2$  and  $\mathfrak{H}_3$ , such that  $\mathfrak{H}_1$  consists of weak identities modulo  $\mathfrak{H}_2$  and  $\mathfrak{H}_2$  consists of weak identities modulo  $\mathfrak{H}_3$ , then it does not follow that  $\mathfrak{H}_1$  consists of weak identities modulo  $\mathfrak{H}_3$ .

*Example 4.6.* Let  $G$  be nonabelian finite simple group. Then as shown in example 2.7, the element  $[g_1, g_2]$  is a weak identity in  $G$  (modulo the trivial verbal subgroup). But the whole free group  $\mathfrak{F}$  consists of weak identities modulo  $\mathfrak{F}'$  because the quotient  $G/[G, G]$  is the trivial group. Notice that  $\mathfrak{F}$  does not consist of weak identities in  $G$  because the group  $G$  is not trivial. This example shows that in general the relation  $\mathfrak{H}_1 \preceq_{wG} \mathfrak{H}_2$  is not transitive.

In order to address this problem we need to define weak\* identities.

*Definition 4.7.* Let  $\mathfrak{S}$  and  $\mathfrak{H}$  be  $T$ -subgroups. We call  $\mathfrak{S}$  weak\* identities modulo  $\mathfrak{H}$ , denoted  $\mathfrak{S} \equiv_w^* 1 \pmod{\mathfrak{H}}$ , if there exist integer  $n$  and  $T$ -subgroups  $\mathfrak{S}_i$ , for  $i = 0, \dots, n$  such that  $\mathfrak{S} = \mathfrak{S}_0$ ,  $\mathfrak{H} = \mathfrak{S}_n$ , and  $\mathfrak{S}_{i-1}$  consists of weak identities modulo  $\mathfrak{S}_i$ , for all  $i = 1, \dots, n$ . The integer  $n$  is called the length of the  $T$ -subgroup  $\mathfrak{S}$  as weak\* identities modulo  $\mathfrak{H}$ .

We say that  $\mathfrak{S}$  consists of weak\* identities in the group  $G$  iff  $\mathfrak{S} \equiv_w^* 1 \pmod{\{1\}}$ , i.e., if they are identities modulo the trivial verbal subgroup.

**Theorem 4.8.** *a) Let  $\mathfrak{H}_i \trianglelefteq_T \mathfrak{F}$ , for  $i = 1, \dots, n$  be a descending chain of  $T$ -subgroups, i.e.,*

$$\mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \dots \supset \mathfrak{H}_{n-1} \supset \mathfrak{H}_n.$$

*Then  $\mathfrak{H}_1 \equiv_w^* 1 \pmod{\mathfrak{H}_k}$  if and only if  $\mathfrak{H}_i \equiv_w^* 1 \pmod{\mathfrak{H}_{i+1}}$  for every  $i = 1, \dots, n-1$ .*

b) Let  $f \in \mathfrak{F}_n$  be a word. Then  $\langle f_{|g_i \rightarrow \mathfrak{S}} \rangle_T$  consists of weak\* identities modulo  $\langle f_{|g_i \rightarrow \mathfrak{H}} \rangle_T$ , provided that  $\mathfrak{S} \equiv_w^* 1(\text{mod } \mathfrak{H})$ .

c) Let  $f \in \mathfrak{F}_n$  be a word and let  $\mathfrak{S}_i$  and  $\mathfrak{H}_i$  be  $T$ -subgroups in  $\mathfrak{F}$ . Then  $\langle f_{|g_i \rightarrow \mathfrak{S}_i} \rangle_T$  consists of weak\* identities modulo  $\langle f_{|g_i \rightarrow \mathfrak{H}_i} \rangle_T$ , whenever  $\mathfrak{S}_i \equiv_w^* 1(\text{mod } \mathfrak{H}_i)$  for every  $i$ .

*Proof.* a) The “if” part follows from the definition of weak\* identities; the “only if” part is trivial.

b), c) Induction on the length of  $\mathfrak{S}$  as a weak\* identity modulo  $\mathfrak{H}$  (the maximal of the length for part c) ) - the base case is trivial, and the induction step follows theorem 4.5.  $\square$

## 5. WEAK\* IDENTITIES IN SOLVABLE GROUPS

The main results in this section are theorems 5.1 and 5.9, which show that in meta-abelian/solvable groups there are many weak\* identities which are not ordinary identities.

**Theorem 5.1.** *If the group  $G$  is finitely generated, then  $\mathfrak{F}'$  consists of weak\* identities in  $G$  modulo  $\mathfrak{F}''$ .*

*Remark 5.2.* The above theorem is not true for an infinitely generated group. Let  $G$  be any group. Then any weak/weak\* identity in the group  $G^{\times\infty}$  is an identity in  $G$  and in  $G^{\times\infty}$  (this is true because the group  $G^{\times\infty}$  is discriminating and we can apply the results from the next section). Therefore, there exists an infinitely generated meta-abelian group  $\tilde{G}$  such that  $[g_1, g_2]$  is not a weak\* identity in  $\tilde{G}$ .

*Proof.* In order to prove the theorem we need to prove several lemmas. Let us fix the finitely generated group  $G$ , generated by the  $d$  elements  $p_1, \dots, p_d$ . Without loss of generality, we may assume that  $G$  is a meta-abelian group.

**Lemma 5.3.** *There exists an integer  $N$ , depending on the group  $G$ , such that the element  $[[g_1, g_2], g_3^N]$  is a weak identity in  $G$  (modulo  $\mathfrak{F}''$ ).*

*Proof.* The group  $G/G'$  acts by conjugation on  $G'$ , and  $G'$  is a finitely generated module over it. Therefore, there exists an integer  $N$ , such that for every  $\bar{g} \in G/G'$  and  $h \in G'$ , the length of the orbit of the element  $h$  under the action of  $\bar{g}$  is either infinity or divides  $N$ . Thus, for every  $g \in G$  and  $h \in G'$ ,  $[h, g^N] \neq 1$  implies that  $[h, g^k] \neq 1$ , for all  $k \neq 0$ .

Suppose that we have a homomorphism  $\rho$  from  $\mathfrak{F}^{\times d+1}$  to  $G$ , such that

$$\rho(i_k([g_1, g_2], g_3^N)) \neq 1, \text{ for all } k.$$

Let  $a_k = \rho(i_k(g_3))$  and  $b_k = \rho(i_k([g_1, g_2]))$ . We have that  $[b_k, a_j^N] \neq 1$  iff  $k = j$ . From the choice of  $N$  we have that  $[b_k, a_k^n] \neq 1$  for all  $n$ . This shows that the element  $a_k^n$  does not lie in the subgroup generated by  $a_j$ , for  $j \neq k$  and  $G'$ , because all elements in this subgroup commute with  $b_k$ . Therefore, the  $a_i$ -es are linearly independent modulo  $G'$ , which is impossible, since the group  $G/G'$  is an abelian group generated by  $d$  elements. Therefore, there is no such homomorphism  $\rho$ , i.e.,  $[[g_1, g_2], g_3^N]$  is a weak identity.  $\square$

**Lemma 5.4.** *The element  $[[g_1, g_2], g_3]$  is a weak identity modulo the  $T$ -subgroup  $\mathfrak{H}_3$  generated by  $\mathfrak{F}''$  and  $[[g_1, g_2], g_3^N]$ .*

*Proof.* Let  $\rho : \mathfrak{F}^{\times n} \rightarrow G$  be a homomorphism, where  $n = d \log_2 N$ . Suppose that  $\rho(i_k([g_1, g_2], g_3)) \notin \mathfrak{H}_3(G)$  for all  $k$ . Let

$$H_k = \{g \in G \mid [g, \rho(i_j([g_1, g_2]))] \in \mathfrak{H}_3(G), \text{ for } j \leq k\}.$$

The groups  $H_k$  form a descending chain of subgroups. The chain is strictly descending, since  $\rho(i_k(g_3)) \in H_{k-1} \setminus H_k$ . All these subgroups contain the subgroup generated by  $G'$  and the elements  $g^N$ , for any  $g \in G$ . Therefore, we can project the subgroups  $H_k$  to subgroups of the group  $G/G' \cdot G^N$ . The last group is a finite group containing less than  $N^d$  elements and does not have a strictly decreasing sequences of subgroups of length more than  $d \log_2 N$  – contradiction. This proves that the element  $[[g_1, g_2], g_3]$  is a weak identity modulo  $\mathfrak{H}_3$ .  $\square$

**Lemma 5.5.** *There exists an integer  $N$ , depending on the group  $G$ , such that the element  $[g_1, g_2^N]$  is a weak identity modulo the  $T$ -subgroup  $\mathfrak{H}_2$  generated by  $[[g_1, g_2], g_3]$ .*

*Proof.* The proof is similar to the one of lemma 5.3. The group  $G'/\mathfrak{H}_2$  is a finitely generated abelian group. Let  $N$  be a number divisible by the order of all torsion elements in this group. This implies that if  $h \in G'$  and  $h^N \notin \mathfrak{H}_2(G)$ , then  $h^k \notin \mathfrak{H}_2(G)$ , for all  $k \neq 0$ . Now if we assume that  $[g_1, g_2^N]$  is not a weak identity, then there exists a map  $\rho : \mathfrak{F}^{\times d+1} \rightarrow G$  such that  $\rho(i_k([g_1, g_2^N])) \notin \mathfrak{H}_2(G)$ . Using arguments similar to the ones in the proof of lemma 5.3, we can show that the elements  $\rho(i_k(g_2))$  are linearly independent modulo  $G'$ , which is impossible – contradiction.  $\square$

**Lemma 5.6.** *The element  $[g_1, g_2]$  is a weak identity modulo the  $T$ -subgroup  $\mathfrak{H}_1$  generated by  $[[g_1, g_2], g_3]$  and  $[g_1, g_2^N]$*

*Proof.* Same as the proof of lemma 5.4, but we use the element  $g_1$ , instead of the element  $[g_1, g_2]$  to construct the groups  $H_k$ .  $\square$

*Remark 5.7.* The statements of Lemmas 5.4 and 5.6 hold in any finitely generated group  $G$ , although the heights of the corresponding weak identities depend only on the number of generators of the group  $G$ . The same is not true for Lemmas 5.3 and 5.5, where the number  $N$  depends on the group  $G$ .

Now we can use these lemmas to prove Theorem 5.1: take the  $T$ -subgroups

$$\mathfrak{F}' \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \mathfrak{H}_3 \supset \mathfrak{F}''.$$

Lemmas 5.3, 5.4, 5.5 and 5.6 show that each of these  $T$ -subgroups consists of weak identities modulo the next one. Therefore, by definition  $\mathfrak{F}'$  consists of weak\* identities modulo  $\mathfrak{F}''$ , which completes the proof of theorem 5.1.  $\square$

*Corollary 5.8.* If for a fixed finitely generated group  $G$ , the  $T$ -subgroup  $\mathfrak{F}'$  consists of weak\* identities modulo  $\mathfrak{F}''$ . Then  $\mathfrak{F}'$  consist of weak\* identities modulo  $\mathfrak{F}^{n'}$  for all  $n$ . Here,  $\mathfrak{F}'$  denotes the commutator subgroup,  $\mathfrak{F}''$  is its commutator subgroup and  $\mathfrak{F}^{n'}$  is the  $n$ -th term of the solvable series of the free group  $\mathfrak{F}$ .

*Proof.* We can use theorem 4.8 c) to show that  $\mathfrak{F}'^k$  consists of weak\* identities in  $G$  modulo the group  $\mathfrak{F}'^{k+1}$ . Finally, using part a) we can show that

$$\mathfrak{F}' \equiv_w^* 1 \pmod{\mathfrak{F}^{n'}}.$$

$\square$

**Theorem 5.9.** *Let  $G$  be a finitely generated solvable group. Then  $\mathfrak{F}'$  consists of weak\* identities.*



*Proof.* Corollary 5.8 and theorem 5.1 imply that  $\mathfrak{F}' \equiv_w^* 1 \pmod{\mathfrak{F}'^n}$ . The group  $G$  is solvable. Therefore,  $\mathfrak{F}'^n(G) = 1$ , and all identities mod  $\mathfrak{F}'^n$  are identities in  $G$ .  $\square$

## 6. DISCRIMINATING GROUPS

Discriminating groups were introduced in [2] and [1] by G. Baumslag, A. Myasnikov and V. Remeslenikov. There the authors consider the universal theory of groups. A group  $G$  is called square-like if the universal theories of  $G$  and  $G \times G$  coincide. There is simple sufficient condition of a group  $G$  to be square-like - a group satisfying this condition is called discriminating. More detailed discussion of discriminating groups can be found in [1], [2] and [3].

A group  $G$  discriminates a group  $H$  if for any finite set of nontrivial elements  $h_i \in H$  there exists a homomorphism  $\phi : H \rightarrow G$  which maps  $h_i$ -es to nontrivial elements in  $G$ . A group  $G$  is called discriminating, if  $G$  discriminates  $G \times G$ . The definition we give below is not exactly the same as the one in [1], but it is equivalent.

**Definition 6.1.** A group  $G$  is called discriminating if for any integer  $N$  and for any elements  $h_1, \dots, h_N \in G \times G$ , there exists a homomorphism  $\rho : G \times G \rightarrow G$  such that  $\rho(h_i) = 1$  if and only if  $h_i = 1$ .

**Lemma 6.2.** *Let  $G$  be a discriminating group then for any integers  $n, N$  and any elements  $h_1, \dots, h_N \in G^{\times n}$  there exists a homomorphism  $\rho : G^{\times n} \rightarrow G$  such that  $\rho(h_i) = 1$  if and only if  $h_i = 1$ .*

*Proof.* We will use induction on  $n$ . The base case  $n = 1$  is trivial (take  $\rho = \text{id}$ ).

Suppose that we have elements  $h_i \in G^{\times n+1}$ . We can express each of them as  $h_i = (h_{i,1}, h_{i,2})$ , where  $h_{i,1} \in G^{\times n}$  and  $h_{i,2} \in G$ . By the induction hypothesis, there exists a homomorphism  $\rho_1 : G^{\times n} \rightarrow G$ , such that  $\rho_1(h_{i,1}) = 1$  iff  $h_{i,1} = 1$ . Now consider the elements  $\bar{h}_i = (\rho_1(h_{i,1}), h_{i,2}) \in G \times G$ . By the discriminating property of the group  $G$ , there exists a map  $\rho_2 : G \times G \rightarrow G$ , such that  $\rho_2(\bar{h}_i) = 1$  iff  $\bar{h}_i = 1$ . The construction of these maps show that if we define  $\rho = \rho_2 \circ (\rho_1 \times \text{id})$ , then  $\rho(h_i) = 1$  iff and only if  $\bar{h}_i = 1$ . This is the same as  $\rho_1(h_{i,1}) = 1$  and  $h_{i,2} = 1$ , which is equivalent to  $h_i = 1$ . This finishes the proof of the induction step and completes the proof of the lemma.  $\square$

**Theorem 6.3.** *Let  $G$  be a discriminating group. If a set  $S$  is a set of weak identities in  $G$ , then every element from  $S$  is an identity in the group  $G$ .*

*Proof.* Assume the contrary: there exists an  $s \in S$  such that  $s$  is not an identity in  $G$ . Then there exists a homomorphism  $\pi$  from  $\mathfrak{F}$  to  $G$ , such that  $\pi(s) \neq 1$ . Let us consider the elements  $h_j = i_j(\pi(s)) \in G^{\times N}$  for  $j = 1, \dots, N$ . By lemma 6.2, there exists a homomorphism  $\tilde{\rho} : G^{\times N} \rightarrow G$ , such that  $\tilde{\rho}(h_j) \neq 1$  for all  $j$ . Finally, define a homomorphism  $\rho : \mathfrak{F}^N \rightarrow G$  by

$$\rho = \tilde{\rho} \circ (\pi \times \pi \times \dots \times \pi).$$

Then by construction we have that  $\rho(i_j(s)) \neq 1$ , which contradicts the fact that  $S$  is a set of weak identities in the group  $G$ .  $\square$

**Remark 6.4.** The converse (that if every weak identity is an identity then the group  $G$  is discriminating) is not true — for example, in any abelian group, all weak identities are identities but not all abelian groups are discriminating.

*Corollary 6.5.* Let  $G$  be a discriminating group. If a  $T$ -subgroup  $\mathfrak{S}$  is a set of weak\* identities in  $G$  modulo the trivial  $T$ -subgroup, then every element from  $\mathfrak{S}$  is an identity in the group  $G$ .

*Proof.* By the definition of weak\* identities, there exists an integer  $n$  and  $T$ -subgroups  $\mathfrak{S}_i$ , for  $i = 0, \dots, n$  such that  $\mathfrak{S} = \mathfrak{S}_0$ ,  $\mathfrak{S}_n = \{1\}$ , and  $\mathfrak{S}_{i-1} \equiv_w 1 \pmod{\mathfrak{S}_i}$ , for all  $i = 1, \dots, n$ . By induction on  $k$ , we can show that  $\mathfrak{S}_{n-k}$  are identities in  $G$ . The induction step is done using theorem 6.3.  $\square$

Apply theorem 6.3 and corollary 6.5 to the results in sections 3 and 5.8 we obtain the following results:

**Theorem 6.6.** *A finitely generated solvable group  $G$  is discriminating if and only if it is isomorphic to  $\mathbb{Z}^n$ .*

*Proof.* By theorem 5.9, the  $T$ -subgroup  $\mathfrak{F}'$  consists of weak\* identities in  $G$ . Now we can apply the previous corollary to show that  $\mathfrak{F}'$  consists of identities in the group  $G$ , i.e., the group  $G$  is abelian. It is easy to see that the only finitely generated abelian groups, which are discriminating, are the torsion free ones.  $\square$

**Theorem 6.7.** *A linear group  $G$  is discriminating only if it is abelian.*

## 7. INTERESTING QUESTIONS

Finally we mention several interesting open questions involving the notion of weak identities.

*Open Question 7.1.* *Is it true that  $\mathfrak{wId}(G)$  is always a set of weak identities in the group  $G$ .*

As mentioned in remark 2.6 this is true if  $\mathfrak{wId}(G)$  is finitely generated as a verbal subgroup. This question is equivalent to the following question: Let  $S$  be a subset of  $\mathfrak{F}$  such that any one element subset of  $S$  is a set of weak identities in  $G$ , is it true that  $S$  is a set of weak identities. The answer is positive if the set  $S$  is finite.

*Open Question 7.2.* *Is it possible to characterize the classes of groups  $\mathfrak{wVar}(\mathfrak{H})$ .*

It can be shown that if a group  $G$  is in the class  $\mathfrak{wVar}(\mathfrak{H})$  and  $\rho : G \rightarrow H$  is surjective homomorphism such that for any finite set of elements  $h_i \in H$  there exist elements  $g_i \in G$  such that  $\rho(g_i) = h_i$  and  $[g_i, g_j] = 1$  whenever  $[h_i, h_j] = 1$ ; then the group  $H$  is also in the class  $\mathfrak{wVar}(\mathfrak{H})$ . This class is closed under taking subgroups, finite Cartesian products and ‘restricted’ homomorphic images.

*Open Question 7.3.* *Describe all group in  $\mathfrak{wVar}(\mathfrak{F}')$ .*

This class contains all groups  $G$  which has bounded centralizer sequences, i.e., all finite groups, all finitely generated meta-abelian groups, all linear groups and all free groups. As mentioned in the previous question, this class is closed under taking subgroups, finite products and restricted homomorphic images. Are there examples of groups  $G \in \mathfrak{wVar}(\mathfrak{F}')$ , which does not have a bounded centralizer sequence?

*Open Question 7.4.* *Describe all finitely generated groups  $G$  such that any weak identity in  $G$  is an identity.*

It is easy to see that for any abelian group  $G$ , every weak identity in  $G$  is an identity. By theorem 6.3 the same is true for any discriminating group. This class is also closed under Cartesian products. Are there examples of non discriminating, non abelian groups, which does not decompose as a Cartesian product, with this property?

*Open Question 7.5. Does there exist a finitely generated group  $G$  such that*

$$\mathfrak{Id}(G) \subset \mathfrak{wId}(G) \subset \mathfrak{F}',$$

*where all the inclusions are strict?*

Any finitely generated solvable group  $G$  such that  $[g_1, g_2]$  is not a weak identity in  $G$  will give positive answer to this question. An interesting generalization is whether there exists a finitely generated group  $G$  such that

$$\mathfrak{Id}(G) \subset \mathfrak{wId}(G) \subset \mathfrak{w}^*\mathfrak{Id}(G) \subset \mathfrak{F}',$$

where all the inclusions are strict.

After completing the work on the paper, the author was informed that Theorems 6.7 and 6.6 were proven independently by A. Myashnikov and P. Shumyatsky [4] using different methods.

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